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ON THE TOPOLOGY OF FOLATIONS

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It is given main authors' results of foliation theory.

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Definition 1. Let (M, A) be a smooth manifold of dimension n, where **A –** C^r atlas. A family $F = \{L_\alpha; \alpha \in B\}$ of path-wise connected subsets of *M* is called *k* – dimensional foliation if it satisfies to following three conditions:

(F₁) $\bigcup_{\alpha \in B} L_\alpha = M$ **;**

(F_{11}) For every $\alpha, \beta \in B$ if $\alpha \neq \beta$, then $L_\alpha \cap L_\beta = Ш$

(F_{111}) For any point $p \in M$ there exists a local chart (local coordinate system) $(U, \phi) \in A,\ p \in U$ so that if $U \cap L_\alpha \neq Ø$ the components of $\phi(U \cap L_\alpha)$ are following subsets of parallel affine planes

$$\left\{ \left(x_1, x_2, ..., x_n \right) \in \phi\left(U \right) : x_{k+1} = c_{k+1},\ x_{k+2} = c_{k+2}, ...,\ x_n = c_n \right\}$$

where numbers $c_{k+1}, c_{k+2}, ..., c_n$ are constant on components.

The most simple examples of a foliation are given by integral curves of a vector field and by level surfaces of differentiable functions.

Using a condition 3 of definition –1 it is easy to establish that there is a differential structure on each leaf such that a leaf is immersed *k*-dimensional submanifold of *M*, i.e the canonical injection is a immersing map (a map of the maximum rank). Thus on each leaf there are two topology: the topology τ_M induced from *M* and it's own topology τ_F as a submanifold. These two topologies are generally different. The topology τ_F is stronger than topology τ_M , i.e. each open subset of L_α in topology τ_M is open in τ_F

The leaf is L_α called as proper if the topology τ_F coincides with the topology τ_M induced from *M*. If these two these topology on L_α do not coincide, the leaf is called as not proper leaf. In work [2] the following assertion is proved which takes place for foliations with singularities too.

Proposition. If a leaf is a closed subset of *M* **then it is a proper leaf.**

Let *L* be a leaf of *F*. Point $y \in M$ is called a limit point of the leaf L if there is a sequence of points y_m from *L* which converges to *y* in topology of manifold *M* and does not converge to this point in the topology of the leaf *L* [2].

Set of all limit points of the leaf *L* we will denote through $\Omega(L)$. $L(y) \subset \Omega(L)$. In work [2] the following is proved

Theorem 1. (1). The leaf L_0 is proper leaf if and only if $L_0 \cap \Omega(L_0) = 0$;

(2). The leaf is L_0 is not proper leaf if and only if $\Omega(L_0) = \bar{L}_0$, where $\bar{L}_0$ – is the closure of L_0 in manifold *M*.

For two leaves L_1 and L_2 we will write in $L_1 \leq L_2$ only in a case when $L_1 \subset \Omega(L_2)$. The inequality $L_1 < L_2$ means $L_1 \leq L_2$ and $L_1 \neq L_2$. The relation $\leq$ on the set of leaves has been entered by T. Nisimori in the paper [3].

We will denote through $(M / F, \leq)$ set of leaves with the entered relation on it. It is obvious that the $\leq$ on M / F reflective and is transitive, but in many cases this relation is not asymmetric, therefore generally the set $(M / F, \leq)$ is not partially ordered.T.Nisimori was interested in the case where $(M / F, \leq)$ is a partially ordered set. Except that T.Nisimori has entered concepts of depth of a leaf *L* and depth of foliation *F* as follows: $d(L) = \sup\{ k |$ there exist leaves $L_1, L_2, ..., L_k$ such that $L_1 < L < ... < L_k = L \}$, $d(F) = \sup\{d(L) | L \in M / F\}$.

The following theorem is proved in paper [2] shows that there exists one dimensional analytical foliation generated by integral curves of analytical vector field which have leaves of depth equal to 1, 2 and 3.

Theorem 2. Let $S^2 \times S$ be k dimensional sphere .On the manifold $S^2 \times S$ there exists an analytical vector field without singular points and with three pairwise different integral curves α, β, γ such that $\alpha \subset \Omega(\beta)$, $\beta \subset \Omega(\gamma)$, where α is a closed trajectory, $\Omega(\beta)$ consists of only closed rajectories, $\Omega(\gamma)$ consists of only the trajectories of depth equal to two.

This vector field generates one-dimensional foliation of the depth equal to 3.

In the paper [3] for codimension one foliation the following theorem is proved:

Theorem 3. If $d(F) < \infty$ or all leaves of foliation *F* are proper,then the set $(M / F, \leq)$ is partially ordered.

Nishimori, studying property codimension one foliation in the case when the set $(M / F, \leq)$ is a partially ordered, has delivered following questions which are of interest for foliation with singularities too [3]:

1. Are all leaves of foliation *F* proper under the assumption that the set $(M / F, \leq)$ is partially ordered?

2. Is a leaf $L -$ proper under the assumption that $dL < \infty$?

A.Narmanov studied the relation $\leq$ for foliation with singularities in the paper [4]. In particular, he proved the following theorems which solves problems 1, 2 delivered by Nishimori.

Theorem 4 [4]. Let M / F be the set of leaves of foliation *F* with singularities. Then the set $(M / F, \leq)$ is a partially ordered if and only if all leaves **are proper.**

Theorem 5 [4]. If the depth of a leaf is finite, then it is proper leaf.

In 1976 in Rio de Janeiro at the international conference the attention to the question on possibility of the proof of theorems on local stability for noncompact leaves has been brought. In 1977 the Japanese mathematician T.Inaba has constructed a counterexample which shows that if codimension of foliation is not equal to one G. Reeb's theorem cannot be generalized for noncompact leaves [18].

Let's bring the theorem on a neighborhood of a leaf with finite depth which is generalization of the theorem of J.Reeb on local stability for transversely oriented codimension one foliation.

Let *F* be a codimension one foliation, *L* be a some leaf of *F* with finite depth, ρ – distance function defined by some fixed riemannian metric on *M*.

Let's enter set $U_r = \{y \in M : \rho(y, L) < r\}, r > 0$, where $\rho(y, L)$ - distance from the point y to the leaf *L*.

Theorem 6 [6]. Let *F* be a transversely oriented codimension one foliation on compact manifold M .If the holonomy pseudogroup Γ the leaf *L* is trivial, for each $r > 0$ there is a invariant open set *V* containing *L* and consisting of leaves diffeomorphic to *L* which satisfies to following conditions:

1) $V \subset U_r$; 2) $dL_\alpha = dL$ for each leaf $L_\alpha \subset V$.

One more generalization of G. Reeb theorem for a noncompact leaf is resulted below.For this purpose we will bring some definitions.

Let L_0 a leaf of codimension one foliation *F*, $U_r = \{x \in M : \rho(x; L_0) < r\}$, where $\rho(x, L_0) -$ distance from the point *x* to a leaf L_0. We will assume that there is such number $r_0 > 0$ that for each horizontal curve $h : [0;1] \to U_{r_0}$ and for each vertical curve $v : [0;1] \to L_0$ such that $v(0) = h(0)$ there exists vertically-horizontal homotopy for pair (v, h). At this assumption we formulate generalization of the theorem of J. Reeb.

Theorem 7 [8]. Let F a transversely oriented codimension one foliation, L_0 be a relatively compact proper leaf leaf with finitely generated fundamental group. Then if holonomy group of the leaf L_0 is trivial then for each $r > 0$ there is an saturated set V such that $L_0 \subset V \subset U_r$ and restriction of F on V is a fibrarion over R^1 with the leaf L_0.

Let M smooth connected complete riemannian manifold. Then on M there are two parallel foliations, mutually additional on orthogonality. If M simple connected manifold then the de Rham theorem takes place which asserts that M is isometric to direct product of any two leaves from different foliations. In this case both foliations are riemannian and total geodesic simultaneously. Below it is presented results of authors on geometry of riemannian and totally geodesic foliations.

Let's assume that F is a riemannian foliation with respect to riemannian metric g on M. Let's denote through $L(p)$ a leaf of F passing through a point p, $F(p)-$ tangent space of leaf at the point p, $H(p)-$ orthogonal complementary of $F(p)$ in T_pM, $p \in M$. There are two subbundle (smooth distributions), $TF = \{F(p) : p \in M\}$, $H = \{H(p) : p \in M\}$ of tangent bundle TM such, that $TM = TF \oplus H$ where H is orthogonal addition TF.

Let $\pi_1 : TM \to TF$, $\pi_2 : TM \to H-$ be orthogonal projections, $V(M)$, $V(F)$, $V(H)$ be the set of smooth sections of bundles TM, TF, H accordingly. Now we will assume that each leaf of F is total geodesic submanifolds of M. It is equivalent to that, $\nabla_X Y \in V(F)$ for all $X, Y \in V(F)$ [1, p.47–61], where $\nabla -$ Levi-Chivita connection. In this case F is a riemannian foliation with total geodesic leaves. Then on bundles TF and H are given metric connections ∇^1 and ∇^2 as follows. If $X \in V(F)$, $Y \in V(H)$, $Z \in V(M)$ we will put $\nabla^1_Z X = \pi_1(\nabla_Z X)$, $\nabla^2_Z Y = \pi_2\left[Z_1, \tilde{Y}\right] + \pi_2\left[\nabla_{Z_2}, Y\right]$, where $Z = Z_1 \oplus Z_2$, $Z_1 \in V(F)$ $Z_2 \in V(H)$, $\tilde{Y} \in V(M)$, $\pi_2\tilde{Y} = Y$. We will intrjduce metric connection $\tilde{\nabla}$ as follows: $\tilde{\nabla}_Z X = \nabla^1_Z X_1 + \nabla^2_Z X_2$, where $X, Z \in V(M)$, $X_i = \pi_i(X)$, $i = 1,2$. It is not difficult to check up that distributions TF and H are parallel with respect to $\tilde{\nabla}$. The following theorem shows that if distribution H is complete integrable, connection $\tilde{\nabla}$ coincides with Levi-Chivita connection ∇.

Theorem 8 [9]. Following assertions are equivalent.

1. Distribution H is complete integrable . 2. $\tilde{\nabla}$ is connection without torsion (i.e $\tilde{\nabla} = \nabla$).

The remark. As shows known Hopf fibration of on three-dimensional sphere, the distribution H it is not always complete integrable.

In the known monograph [1] Ph. Tondeur studied function $f : M \to R^1$ without critical points on Riemannian manifold M for which length of a gradient is constant on each level surface. For such functions he has proved that Foliation generated by level surfaces of such function, is a Riemannian Foliation.

Definition 1.2. Let M – smooth manifold of dimension n. Function $f : M \to R^1$ of the class $C^2\left(M, R^1\right)$ for which length of a gradient is constant on connection components of level sets is called a metric function.

The following theorem gives complete classification of foliations generated by level surfaces of metric function [7].

Theorem 9 [7]. Let $f : M \to R^1$ be a metric function given in R^n. Then level surfaces of function f form foliation which has one of following n types:

Foliation F consists of parallel hyperplanes;2) Foliation F consists of concentric hyperspheres and the point (the center of hyperspheres);3) Foliation F consists of concentric cylinders of the kind $S^{n-k-1} \times R^k$ and the singular leaf R^k (which arises at degeneration of spheres to a point), where k – the minimum of dimensions of critical level surfaces, $1 \le k \le n-2$.

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